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Casimir forces in a plasma: possible connections to Yukawa potentials

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Abstract. We present theoretical and numerical results for the screened Casimir effect between perfect metal surfaces in a plasma. We show how the Casimir effect in an electron-positron plasma can provide an important contribution to nuclear interactions. Our results suggest that there is a connection between Casimir forces and nucleon forces mediated by mesons. Correct nuclear energies and meson masses appear to emerge naturally from the screened Casimir-Lifshitz effect.

1 Introduction

The Casimir and Lifshitz theories of intermolecular (dispersion) forces [1–3] have occupied such a vast literature that little should remain to be said [4–8]. However, there exist still many gaps in our knowledge of the theory of dispersion forces. For instance, we will show in this paper that the presence of any non-zero plasma density between two perfectly reflecting plates fundamentally alters their long-range Casimir interaction. Such finite plasma densities are always present near metal surfaces. These results are discussed in detail in Section 2 where we give theoretical and numerical results for the Casimir interaction between two perfect metal surfaces in the presence of a plasma.

The importance of Casimir forces for electron stability [9–13], particle physics, and in nuclear interactions [14], has been predicted over the years. The problem we intend to revisit is similar in spirit to the old story called "the Casimir mousetrap" for the stability of charged electrons [10,13]. The negative charges on an electron surface give rise to a repulsive force between the different parts of the surface that has to be counteracted by an attractive force in order for the electron to have a finite radius. Casimir proposed that such attractive Poincaré stresses could come from the zero-point energy of electromagnetic vacuum fluctuations [9]. A number of attempts have been made to compute such Casimir energies [10–13]. However, all concluded that while the magnitude of the interaction was correct, it had the wrong sign. Further it gave a repulsive force [10–13].

Finite plasma densities are present between nuclear particles due to the presence of the plasma of the fluctuating electron-positron pairs constantly created and annihilated. The magnitude and asymptotic form of the screened Casimir potential between reflecting surfaces in the presence of this electron-positron plasma suggest a possible connection between Casimir forces and nucleon forces [14]. In Section 3 we proceed to explore this intriguing similarities of the screened Casimir potential with the Yukawa potential for nuclear particles as mediated by mesons. Essentially correct nuclear energies, meson masses and meson lifetimes appear to emerge naturally from the Casimir-Lifshitz theory. When taken at face value, the screened-Casimir model of the Yukawa potential would offer an alternative explanation of nuclear forces as being due to virtual electron-positron excitations.

A somewhat complementary effect is the Casimir force due to electronic wave-function overlap as discussed in reference [15]. In the latter case, the force results from real plate electrons whose evanescent wave functions exponentially decay into the gap between the plates. On the contrary, in our scenario virtual electron-positron pairs in the space between the plates mediate the force.

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2 Casimir effect between perfect metal surfaces in the presence of a plasma

Consider the Casimir-Lifshitz interaction between ideal metal surfaces separated by a plasma of dielectric permittivity

$$\varepsilon(i\omega) = 1 + \frac{4\pi\rho e^2}{m\omega^2} = 1 + \frac{\omega_p^2}{\omega^2}, \tag{1}$$

where the plasma frequency is identified as $\omega_p^2 = 4\pi\rho e^2/m$, ρ is the number density of the plasma, m the electron mass, and e the unit charge. We define some additional variables $\bar{\rho} = \rho e^2\hbar^2/\left(\pi m k^2 T^2\right)$, $\kappa = \omega_p/c$ (note the occurrence of a factor mc^2 in the screening parameter κ), and $x = 2kTl/(\hbar c)$. In these expressions k is Boltzmann's constant, $\hbar$ is Planck's constant, T the effective temperature of the plasma, c is the velocity of light, and l the distance between the plates. The exact expressions for the Casimir-Lifshitz free energy between both real and perfect metal surfaces across a plasma are given in Appendix A. We have found (see Appendix B for a derivation) that the asymptotic interaction energy can at high temperatures and/or large separations be written as:

$$F(l,T) = F_{n=0} + F_{n>0}, \tag{2}$$

$$F_{n=0}(l,T) \approx -\frac{kT\kappa^2}{2\pi}e^{-2l\kappa}\left[\frac{1}{2l\kappa} + \frac{1}{4l^2\kappa^2}\right], \tag{3}$$

$$F_{n>0} \approx \frac{(kT)^2}{l\hbar c}e^{-\pi\bar{\rho}x}e^{-2\pi x} + O\left(e^{-x^2}\right). \tag{4}$$

Here we have separated the zero and finite frequency contributions. These expressions may be useful for theoretical comparisons with experimentally measured Casimir-Lifshitz forces [6,16–25] between metal surfaces interacting across a high density plasma.

We first recall the present understanding of Casimir effect between real metal surfaces in the absence of any intervening plasma. Figure 1 shows the experimental result of Lamoreaux [16,17], compared to the theoretical results of Boström and Sernelius [18]. All curves show the interaction energy divided by the result of the ideal Casimir gedanken experiment at zero temperature, $-\hbar c\pi^2/\left(720d^3\right)$. The lowest curve is for gold at room temperature. It was derived using tabulated optical data for gold as input. Use of the Drude model gives overlapping results. To be noted is that theory and experiment clearly disagree for the cluster of experimental points around $d = 1$ μm. The experimental results agree better with the zero-temperature results (upper solid curve) and even with the zero- or finite-temperatures results for ideal metals (the Casimir gedanken experiment, dotted curves). The agreement is even better with the theoretical room temperature result obtained when using the plasma model.

This puzzling behavior has given rise to a long-standing controversy in the field. We note that the zero frequency part of the Casimir interaction between real metal surfaces depends on how the dielectric function of

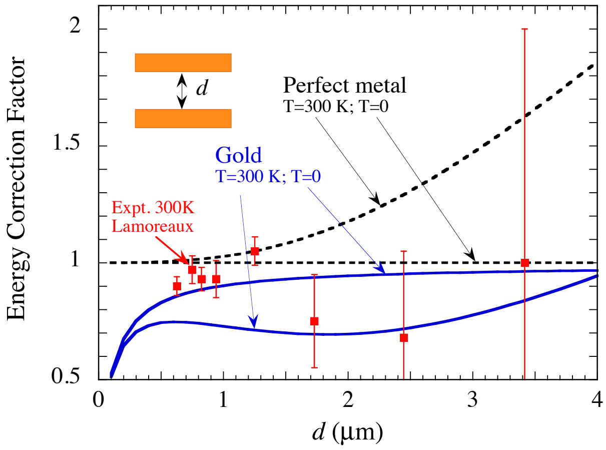

Fig. 1. Energy correction factor for two gold plates in the absence of any intervening plasma. The filled squares with error bars are the Lamoreaux' experimental [16,17] values from the torsion pendulum experiment. The dashed curves are the perfect metal results. The thick solid curves are the results for real gold plates at zero temperature and at room temperature [18]. The dielectric properties for gold was obtained from tabulated experimental optical data.

the metal surfaces is treated. Different theoretical groups have found very different results [18–22]. A most valuable property of the Lamoreaux experiment [16,17] was that it was carried out at large separations. Lamoreaux was also involved in a more recent version of his old experiment [23] (cf. also the comments of Milton [7]), where plate separations between 0.7 and 7 μm were tested. Quite convincingly, the theoretical predictions based upon the Drude model were found to agree with the observed results to high accuracy.

The thermal Casimir effect is however a many-facetted phenomenon and care has to be taken about the electrostatic patch potentials, which cause uncertainties in the interpretation of the data in the mentioned experiment. There are other experiments, in particular the very accurate one of Decca et al. [26], which yield results apparently in accordance with the plasma model rather than the Drude model. The reason for this conflict between experimental results is not known in the community. It has been suggested occasionally that it might have something to do with the so-called Debye shielding, which can change the effective gap between plates from the geometrically measured width. But the experimentalists themselves turn out to be skeptical towards such a possibility (an elementary overview of the temperature dependence of the Casimir force is recently given in Ref. [27]). There is clearly an urgent need for more experiments and theoretical analysis focusing on Casimir-Lifshitz forces in different systems that include metal surfaces.

As we have shown so far, the presence of any intervening plasma is of importance for the long range interaction energy. We explore next the effect on the energy correction factor for different plasma densities between two ideal surfaces (see Fig. 2). Again, all curves show the interaction energy divided by the result of the ideal Casimir

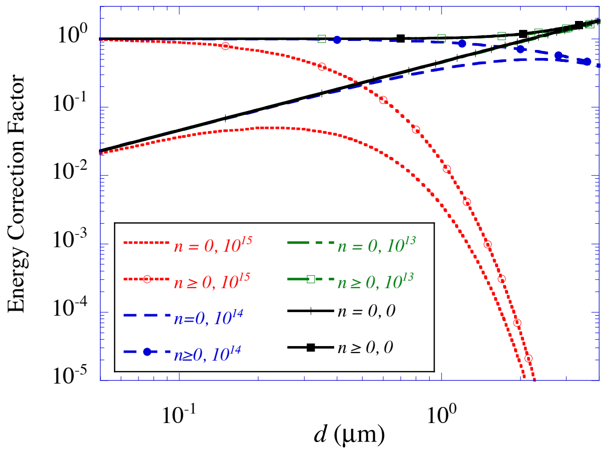

Fig. 2. Energy correction factor for two perfect metal plates interacting across a plasma. The curves are the results for perfect metal plates at room temperature for different plasma frequencies (ω_p in units of rad/s) for the intervening plasma. We show the results for the $n = 0$ and $n \geq 0$ contributions to the interaction free energy.

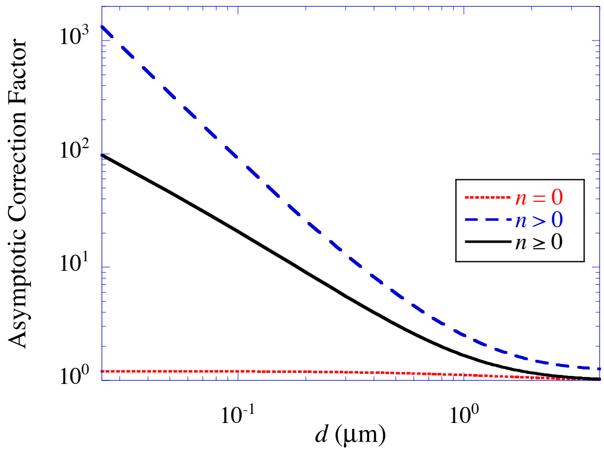

Fig. 3. Asymptotic correction factor for two perfect metal plates interacting across a plasma ($\omega_p = 10^{14}$ rad/s). The results show the ratio between numerically calculated energies and the corresponding asymptotes given in the text. There is very good agreement (ratio close to one) for the $n = 0$ contribution in the entire range considered. For $n > 0$ and $n \geq 0$ the curves go towards one at large surface separations. Note that the asymptotes become more accurate for higher plasma densities.

gedanken experiment at zero temperature in the absence of a plasma. At large separation the result is strongly influenced by intervening plasma, leading to a considerable reduction of the interaction energy. The results show that even weak intervening plasmas can strongly affect Casimir force measurements. The possible presence of spurious plasma densities thus has to be considered carefully.

We next investigate the accuracy of the asymptotes (3) and (4) by comparing their predictions with exact numerical results.

Figure 3 shows the ratio between numerically calculated free energy between two perfect metal plates across a plasma ($\omega_p = 10^{14}$ rad/s) to the corresponding asymptotes given in the text. We see that in this case the asymptote for the $n = 0$ term is very accurate. For the $n > 0$ and $n \geq 0$ contributions this ratio only goes towards one at large separations. The asymptotes become better for higher plasma densities.

3 A contribution from screened Casimir interaction in nuclear interactions

We will now point out a potential connection with the meson theory. That is, if we take the Casimir expansion without a plasma, the first three terms (see Eq. (7) below) are: (1) the usual zero-point fluctuation energy (also equivalent to current-current correlations); (2) a "chemical potential" term, identifiable as the energy of an electron positron pair sea (see Landau and Lifshitz [28]); (3) the black body radiation in the gap. One can then ask how electromagnetic (EM) theory can give rise to weak interactions of particle physics. Such contribution from EM theory comes out if one equates the zero-point energy to the black body radiation term. That gives an equivalent density for the electron positron pair sea and the energy of interaction of about 8 MeV. This agrees with the experimentally found nuclear interaction energy. The form of the interaction with a plasma in the gap is the same as that for the Klein-Gordon-Yukawa potential with the plasma excitation corresponding to and identical with the π_0 meson mass (this assumes a plate size of 1 fermi squared in area and that the planar results translated roughly over to that for spheres).

Now we will explore these ideas in more detail. The screened Casimir free energy asymptotes in the previous section can be compared with the Yukawa potential between nuclear particles at distances large compared with the screening length $l_\pi = \hbar/m_\pi c$ (m_π is the mass of the mediating meson),

$$F(l, T) \propto e^{-l/l_\pi}. \tag{5}$$

To test if the idea can be correct, we first extract the meson mass by taking the exponents in the $F_{n=0}$ term given in the previous section and the Yukawa potential to be equal:

$$m_\pi = \frac{4e\hbar}{c^2}\sqrt{\frac{\pi(\rho_+ + \rho_-)}{m}}. \tag{6}$$

Since we know the meson mass (135 MeV) we estimate the screening length to be 1.458 fm and we also find the density of electrons and positrons that would be required to generate this Yukawa potential from the Casimir effect. The equilibrium of electron positron production can at high temperatures be written as $\rho_\pm = 3\zeta(3)k^3T^3/(2\pi^2\hbar^3c^3)$ [28]. This means that the required effective temperature of nuclear interaction via a screened Casimir interaction is 3.2×10^{11} K.

We now address the important question where the energy to generate this local electron plasma can come from [14]. Feynman speculated that high energy potentials could excite states corresponding to other eigenvalues, possibly thereby corresponding to different masses [29]. It turns out that the low-temperature Casimir interaction, i.e., without an intervening plasma, by itself could be capable of generating the effective temperature required to obtain the plasma. The connection between temperature and density of electrons and positrons given above is exploited in the expression for low temperature Casimir interaction between perfectly reflecting surfaces. In the absence of an intervening plasma, it can be written as equation (35) in reference [30]:

$$F(l,T) \approx \frac{-\pi^2 \hbar c}{720 l^3} - \frac{\zeta(3) k^3 T^3}{2\pi \hbar^2 c^2} + \frac{\pi^2 l k^4 T^4}{45 \hbar^3 c^3}. \qquad (7)$$

This can further be re-written as:

$$F(l,T) \approx \frac{-\pi^2 \hbar c}{720 l^3} - \frac{\pi(\rho_- + \rho_+)\hbar c}{6} + \frac{\pi^2 l k^4 T^4}{45 \hbar^3 c^3}, \qquad (8)$$

where the first term is the zero-temperature Casimir energy, the third is the blackbody energy, and the second has been rewritten in terms of electron and positron densities. If we assume that the entire zero-temperature Casimir energy is transformed into blackbody energy (which at high temperatures can generate an electron-positron plasma) we can estimate the temperature as $T \approx \hbar c/2lk$. This will at a distance of 3.6 fermi give the required effective temperature (at the distances discussed above the effective temperature is even larger, around 2.3×10^{12} K). It is intriguing that a cancellation of the Casimir zero-point energy and the blackbody energy term, just like the cancellation of the $n = 0$ term at low temperatures, gives the right result.

The screened Casimir interaction between two perfectly reflecting surfaces, with estimated cross section of 1 fermi squared a distance 0.5 fermi apart, receives around 4.25 MeV from the $n = 0$ term and 3.25 MeV from the $n > 0$ terms. While the screening length of the $n = 0$ term is defined above we find that the screening length of the $n > 0$ terms also comes out of the right order of magnitude (it is within the crude approximations made of the order one fermi). The nuclear interaction as a screened Casimir interaction would thus receive approximately equal contributions from the classical ($n = 0$) and quantum ($n > 0$) terms. The result compares remarkably well with the binding energy of nuclear interaction that is around 8 MeV.

If the arguments we have given connecting nuclear and electromagnetic interactions have any substance, it is hard to avoid the speculation that the standard decomposition in nuclear physics into coulomb and nuclear force contributions may not be entirely correct. In the insightful words of Dyson: "the future theory will be built, first of all upon the results of future experiments, and secondly upon an understanding of the interrelations between electrodynamics and mesonic and nucleonic phenomena" [31]. The problem is precisely equivalent to that which occurs in physical chemistry where standard theories have all been based on the ansatz that electrostatic forces (treated in a nonlinear theory) and electrodynamic forces (treated in linear approximation by Lifshitz theory) are separable. The ansatz violates both the Gibbs adsorption equation and the gauge condition on the electromagnetic field [32]. When the defects are remedied a great deal of confusion appears to fall into place. If these results are not acceptable within the standard model one must still consider the presence of this additional electromagnetic fluctutation interaction energy between nuclear particles.

4 Conclusions

We have explored the effect of an intervening plasma on the Casimir force between two perfectly conducting plates. The analytically derived asymptotes for large plate separations show that even spurious plasma densities can considerably reduce the expected Casimir force. The effect of plasmas should therefore carefully be considered in Casimir force measurements.

In addition, the derived asymptotes show an interesting structural analogy with the Yukawa potential of nuclear interactions. We have explored this analogy to discuss whether the electromagnetic Casimir effect can possibly explain these interactions. The comparison yields predictions for the required virtual electron-positron plasma density which, however, is only achievable at very large ambient temperatures. If the potential connection to nuclear interactions is correct, then we speculate that the charged π_+ and π_- mesons would come out to be bound positron-plasmon and electron-plasmon excitations in the electron-positron plasma.

Apart from these speculations, our main idea has been to investigate to what extent the screened Casimir effect between perfect metal surfaces, intervened by an electron-positron plasma, can be applied to estimate nucleon forces mediated by mesons. Figures 2 and 3 show the effect of plasma screening; especially the large suppression of the Casimir energy when the plasma density is large, is clearly shown in Figure 2. Our main findings are that nuclear energies and meson masses emerge numerically of the right order of magnitude, thus indicating that our basic idea is a viable one.

Of course, the ideas explored in this paper are somewhat speculative. In principle, although the Casimir energy has the right order of magnitude to provide the required temperature, one may object that it is not evident how this energy can be converted to thermal radiation. The point we wish to emphasize here is that the present arguments, although incomplete, may serve as a useful starting point for further research in this direction, perhaps within the framework of quantum statistical mechanics.

A final comment: use of the electrodynamic Casimir effect in the context of nuclear physics is of course not new. For instance, in hadron spectroscopy viewed from the standpoint of the MIT quark bag model it has long been known that the zero-point fluctuations of the quark and gluon fields may generate a finite zero-point energy of the form $E_{\mathrm{zp}} = -Z_0/r$, for massless quarks. The constant Z_0

is not firmly grounded theoretically, but serves a phenomenological term fitting the experimental data (a classic review article in this field is that of Hasenfratz and Kuti [33]). The phenomenological quark model in which the r-dependent part $\Delta m(r)$ of the effective quark mass $m(r)$ varies according to a Gaussian, $\Delta m(r) \propto -e^{-r^2/R_0^2}$, can also be regarded as an example of essentially the same kind [34].

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Appendix A: Casimir-Lifshitz free energy

One way to find retarded van der Waals or Casimir-Lifshitz interactions between two objects interacting across a medium is in terms of the electromagnetic normal modes of the system [36,37]. For planar structures the interaction energy per unit area can be written as:

$$E = \hbar \int \frac{d^2q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{2\pi} \ln\left[f_q\left(i\omega\right)\right], \tag{A.1}$$

where f_q is the mode condition function with $f_q\left(\omega_q\right) = 0$ defining electromagnetic normal modes. Equation (A.1) is valid for zero temperature and the interaction energy is the internal energy. At finite temperature the interaction energy is a free energy and can be written as:

$$F = \sum_{n=0}^{\infty} F_n = \frac{1}{\beta} \int \frac{d^2q}{(2\pi)^2} \sum_{n=0}^{\infty}{}' \ln\left[f_q\left(i\xi_n\right)\right], \tag{A.2}$$

where $\beta = 1/kT$, and the prime on the summation sign indicates that the term for $n = 0$ should be divided by two. The integral over frequency in equation (A.1) has been replaced by a summation over discrete Matsubara frequencies

$$\xi_n = \frac{2\pi n}{\hbar\beta};\ n = 0,\, 1,\, 2,\, \ldots \tag{A.3}$$

For planar structures the quantum number that characterizes the normal modes is $\mathbf{q}$, the two-dimensional (2D) wave vector in the plane of the interfaces. Two mode types can occur: transverse magnetic (TM) and transverse electric (TE). These dictate the form through the wave amplitude reflection coefficients, r. For instance, for two planar objects in a medium, corresponding to the geometry 1|2|1, the mode condition function is given by:

$$f_q = 1 - e^{-2\gamma_2 d} {r_{12}}^2, \tag{A.4}$$

where d is the thickness of intermediate medium, and the reflection coefficients for a wave impinging on an interface between medium 1 and 2 from the 1-side given as:

$$r_{12}^{\mathrm{TM}} = \frac{\varepsilon_2\gamma_1 - \varepsilon_1\gamma_2}{\varepsilon_2\gamma_1 + \varepsilon_1\gamma_2}, \tag{A.5}$$

and

$$r_{12}^{\mathrm{TE}} = \frac{(\gamma_1 - \gamma_2)}{(\gamma_1 + \gamma_2)}, \tag{A.6}$$

for TM and TE modes, respectively. Here, γ_j stands for

$$\gamma_i(\omega) = \sqrt{q^2 - \varepsilon_i\left(\omega\right)\left(\omega/c\right)^2}, \tag{A.7}$$

where $\varepsilon_i\left(\omega\right)$ is the dielectric function of medium i, and c the speed of light in vacuum.

For two perfectly conducting plates ($r_{12}^{\mathrm{TE}} = -1$, $r_{12}^{\mathrm{TM}} = 1$), the Casimir energy (A.2) across a dissipation-free plasma takes the simple form

$$F(l,T) = \frac{kT}{\pi} \sum_{n=0}^{\infty}{}' \int_0^\infty dqq \ln\left[1 - e^{-2l\sqrt{q^2 + (\xi_n/c)^2 + \kappa^2}}\right], \tag{A.8}$$

recall that $\kappa = \omega_p/c$. By a simple variable substitution, the first term in the Matsubara sum can be cast into the alternative form

$$F_{n\,=\,0}(l,T) = \frac{kT}{2\pi} \int_\kappa^\infty dtt \ln\left(1 - e^{-2lt}\right). \tag{A.9}$$

Appendix B: Asymptotic Casimir free energy in a plasma

Exact treatments of Casimir forces between perfect metal surfaces across a plasma using the above expressions typically give asymptotic expansions that are not uniformly valid. The treatment we present here gives a different result. Our starting point is the above formula (A.8) for the interaction of two perfectly conducting plates accross an intervening plasma as re-written by Ninham and Daicic [30,37,38].

$$F(l,T) = -\frac{kT}{4\pi l^2} \frac{1}{2\pi i} \int_c ds \frac{\Gamma(s)\zeta(s+1)}{(s-2)(2\pi x)^{s-2}} \times \sum_{n=0}^{\infty}{}' (n^2 + \bar{\rho})^{1-s/2}. \tag{B.1}$$

The zero frequency ($n = 0$) term gives the following contribution,

$$F_{n=0}(l,T) = -\frac{kT}{8\pi l^2} \frac{1}{2\pi i} \int_c ds \frac{\Gamma(s)\zeta(s+1)}{(s-2)(2\kappa l)^{s-2}}, \tag{B.2}$$

$$F_{n=0}(l,T) = \frac{kT}{2\pi} \int_\kappa^\infty dtt \ln\left(1 - e^{-2lt}\right), \tag{B.3}$$

which at large separations becomes [30]:

$$F_{n=0} \approx -\frac{kT\kappa^2}{2\pi} e^{-2l\kappa} \left[\frac{1}{2l\kappa} + \frac{1}{4l^2\kappa^2}\right]. \tag{B.4}$$

The interaction free energy can be written as [30]:

$$F(l,T) = -\frac{kT}{8\pi l^2}\frac{1}{2\pi i}\int_c ds \frac{\Gamma(s)\zeta(s+1)}{(s-2)(2\pi x)^{s-2}} \times \zeta_G(-1+s/2,\bar{\rho}), c>3, \quad (B.5)$$

$$\zeta_G(z,a) = 2\zeta_{EH}(z,a) + a^{-z} = 2\sum_{n=1}^{\infty}\frac{1}{(n^2+a)^z} + a^{-z} \quad (B.6)$$

where as discussed in detail by Ninham and Daicic the generalized Epstein-Hurwitz ζ function ζ_G is meromorphic and has simple poles in the complex plane at $z=-k+1/2$ $(k=0,1,2,\dots)$ [30]. In the limit of low temperatures or distances $x \ll 1$ they found (see Ref. [30] for the complete expression)

$$F(l,T) = \frac{-\pi\hbar c}{720 l^3}\left[1 - 15\frac{\rho e^2\hbar^2}{(\pi m k^2 T^2)}\left(\frac{2kTl}{\hbar c}\right)^2 - \dots\right] \quad (B.7)$$

where at low temperatures the $n=0$ term cancels out a contribution from the higher frequency terms.

It is possible with some algebra to express the Lifshitz free energy between two ideal metal plates with intervening plasma in the following form (useful for deriving the asymptotes considered in this contribution):

$$F(l,T) = -\frac{kT}{4\pi l^2}\eta(l,T) = \frac{-kT}{4\pi l^2}\left[\pi x^3 I(\bar{\rho},x)\right]. \quad (B.8)$$

The integral I consists of two parts,

$$I = I_1 + I_2, \quad (B.9)$$

where

$$I_1 = \int_0^{\infty} dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}\left(x^2/y\right), \quad (B.10)$$

and

$$I_2 = \int_0^{\infty} dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}\left(x^2/y\right) 2\bar{\omega}(y), \quad (B.11)$$

respectively.

The function $\bar{\omega}(y)$ appearing in both integrands is defined as [38],

$$\bar{\omega}(y) \equiv \sum_{n=1}^{\infty} e^{-n^2\pi y} \equiv \frac{1}{2}\left\{-1 + y^{-1/2}\left[1+2\bar{\omega}(1/y)\right]\right\}. \quad (B.12)$$

The sum converges faster the larger the y-value. To make use of this fact we divide the integration range for I_1 into two parts and use the two different expressions for the sum in the two resulting integrals. Thus,

$$I_1 = H_1 + H_2, \quad (B.13)$$

where

$$H_1 = \int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} y^{-5/2} \times \left[\tfrac{1}{2}\left(-1+\sqrt{y/x^2}\right) + \sqrt{y/x^2}\,\bar{\omega}\left(y/x^2\right)\right], \quad (B.14)$$

and

$$H_2 = \int_0^{x^2} dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}\left(x^2/y\right), \quad (B.15)$$

respectively.

The integrand in I_2 has a product of two $\bar{\omega}$ functions with different arguments. Here we divide the integration range into three regions and choose the form of the sum that gives the fastest convergence. Under the assumption that $x>1$ we have:

$$\begin{aligned} I_2 &= \int_0^1 dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}\left(x^2/y\right) \times \left[-1 + y^{-1/2}\left(1+2\bar{\omega}(1/y)\right)\right] \\ &+ 2\int_1^{x^2} dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}\left(x^2/y\right)\bar{\omega}(y) \\ &+ \int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}(y) \times \left[-1+\sqrt{y/x^2}\left(1+2\bar{\omega}\left(y/x^2\right)\right)\right] \\ &= J_1+J_2+J_3+J_4+J_5+J_6+J_7, \end{aligned} \quad (B.16)$$

where

$$J_1 = 2\int_1^{x^2} dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}\left(x^2/y\right)\bar{\omega}(y), \quad (B.17)$$

$$J_2 = \frac{2}{x}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} y^{-2}\bar{\omega}(y)\bar{\omega}\left(y/x^2\right), \quad (B.18)$$

$$J_3 = 2\int_0^1 dy y^{-3}\bar{\omega}(1/y)\bar{\omega}\left(x^2/y\right) e^{-\pi\bar{\rho}y}, \quad (B.19)$$

$$J_4 = -\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}(y), \quad (B.20)$$

$$J_5 = \frac{1}{x}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} y^{-2}\bar{\omega}(y), \quad (B.21)$$

$$J_6 = -\int_0^1 dy e^{-\pi\bar{\rho}y} y^{-5/2}\bar{\omega}\left(x^2/y\right), \quad (B.22)$$

and

$$J_7 = \int_0^1 dy e^{-\pi\bar{\rho}y} y^{-3}\bar{\omega}\left(x^2/y\right), \quad (B.23)$$

respectively.

We now add the two I terms and recombine the integral terms to find:

$$I = I_1+I_2 = J_1+J_2+J_3+K_1+K_2+K_3+K_4+K_5, \quad (B.24)$$

where

$$K_1 = \frac{1}{x}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} y^{-2}\left[\frac{1}{2}+\bar{\omega}(y)\right], \quad (B.25)$$

$$K_2 = \frac{-1}{x}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} x y^{-5/2}\left[\frac{1}{2}+\bar{\omega}(y)\right], \quad (B.26)$$

$$K_3 = J_7, \quad (B.27)$$

$$K_4 = \frac{1}{x}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y} y^{-2}\bar{\omega}(y/x^2), \quad (B.28)$$

and

$$K_5 = \int_1^{x^2} dy e^{-\pi\bar{\rho}y} y^{-5/2} \bar{\omega}\left(x^2/y\right)$$
$$= \frac{1}{x^3}\int_1^{x^2} dy e^{-\pi\bar{\rho}x^2/y}\sqrt{y}\bar{\omega}(y), \tag{B.29}$$

respectively.

In K_4 we let $y \to x^2\xi$,

$$K_4 = \frac{1}{x^3}\int_1^{\infty} d\xi e^{-\pi\bar{\rho}x^2\xi}\xi^{-2}\bar{\omega}(\xi), \tag{B.30}$$

and let $\xi \to 1/y$, use the definition of $\bar{\omega}(y)$, and separate into three terms,

$$K_4 = M_1 + M_2 + M_3 \tag{B.31}$$

where

$$M_1 = \frac{-1}{2x^3}\int_0^1 dy e^{-\pi\bar{\rho}x^2/y} = \frac{-1}{2x}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y}y^{-2}, \tag{B.32}$$

$$M_2 = \frac{1}{2x^3}\int_0^1 dy\sqrt{y}e^{-\pi\bar{\rho}x^2/y} = \frac{1}{2}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y}y^{-5/2}, \tag{B.33}$$

and

$$M_3 = \frac{1}{x^3}\int_0^1 dy\sqrt{y}e^{-\pi\bar{\rho}x^2/y}\bar{\omega}\left(y\right), \tag{B.34}$$

respectively.

M_1 and M_2 exactly cancel with the terms with $1/2$ in K_1 and K_2. Now we combine the expressions for K_4 and K_5 and insert into the expression for I,

$$I = I_1 + I_2 = J_1 + J_2 + J_3 + K_3 + N_1 + N_2, \tag{B.35}$$

where

$$N_1 = \frac{1}{x}\int_{x^2}^{\infty} dy e^{-\pi\bar{\rho}y}(y^{-2} - xy^{-5/2})\bar{\omega}(y), \tag{B.36}$$

and

$$N_2 = \frac{1}{x^3}\int_0^{x^2} dy e^{-\pi\bar{\rho}x^2/y}\sqrt{y}\bar{\omega}(y), \tag{B.37}$$

respectively.

Of these terms J_2, J_3, K_3 and the term $\int_{x^2}^{\infty}$ are all $O(e^{-x^2})$ and we may drop them. So we have apart from a term $O(e^{-x^2})$ the following expressions:

$$\eta(l,T) = \eta_1 + \eta_2 = \pi x^3(I_1 + I_2), \tag{B.38}$$

where

$$\eta_1 \approx \pi\int_0^{x^2} dy\sqrt{y}e^{-\pi\bar{\rho}x^2/y}\bar{\omega}(y), \tag{B.39}$$

and since

$$\eta_1 \approx \pi\int_0^{\infty} dy\sqrt{y}e^{-\pi\bar{\rho}x^2/y}\bar{\omega}(y) - O\left(e^{-x^2}\right), \tag{B.40}$$

we may write

$$\eta_1 \approx \pi\int_0^{\infty} dy\sqrt{y}e^{-\pi\bar{\rho}x^2/y}\bar{\omega}(y). \tag{B.41}$$

Using the definition of $\bar{\omega}(y)$ and the following representation:

$$e^{-y} = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} dp y^{-p}\Gamma(p), \mathrm{Re}(p) = C > 0, \tag{B.42}$$

we obtain

$$\eta_1 \approx \pi\int_0^{\infty} dy\sqrt{y}\int_{C-i\infty}^{C+i\infty} dp\sum_{n=1}^{\infty}\frac{\Gamma(p)}{(n^2\pi y)^p}e^{-\pi\bar{\rho}x^2/y}. \tag{B.43}$$

With a variable substitution

$$\kappa^2 l^2 = \pi^2\bar{\rho}x^2 = 4\pi\rho e^2 l^2/\left(mc^2\right),$$

we find

$$\eta_1 \approx \frac{1}{\sqrt{\pi}}\int_0^{\infty} dy\sqrt{y}\int_{C-i\infty}^{C+i\infty} dp\sum_{n=1}^{\infty}\frac{\Gamma(p)}{(n^2 y)^p}e^{-\kappa^2 l^2/y} \tag{B.44}$$

and using the Riemann ζ function,

$$\eta_1 \approx \frac{1}{\sqrt{\pi}}\int_{C-i\infty}^{C+i\infty} dp\Gamma(p)\zeta(2p)\frac{(\kappa l)^3}{(\kappa l)^{2p}}\int_0^{\infty} dy y^{p-5/2}e^{-y}. \tag{B.45}$$

Integration over y results in

$$\eta_1 \approx \frac{(\kappa l)^3}{\sqrt{\pi}}\int_{C-i\infty}^{C+i\infty} dp\frac{\Gamma(p)\zeta(2p)}{(\kappa l)^{2p}}\Gamma(p-\tfrac{3}{2}), \mathrm{Re}(p) = C > \tfrac{3}{2}. \tag{B.46}$$

We now exploit relations for the Γ function:

$$\Gamma(p-3/2) = \frac{\Gamma(p+1/2)}{(p-1/2)(p-3/2)}, \tag{B.47}$$

and

$$\Gamma(p)\Gamma(p+1/2) = \sqrt{\pi}2^{1-2p}\Gamma(2p), \tag{B.48}$$

to obtain

$$\begin{aligned}
\eta_1 &\approx 4(\kappa l)^2 \int_{c-i\infty}^{c+i\infty} dp \frac{\Gamma(p)\zeta(p+1)}{(2\kappa l)^p (p-2)} \\
&= 4(\kappa l)^2 \int_{c-i\infty}^{c+i\infty} dp \Gamma(p) \sum_{n=1}^{\infty} \frac{1}{n^{p+1}(2\kappa l)^p (p-2)} \\
&= 2(\kappa l)^2 \int_{c-i\infty}^{c+i\infty} dp \Gamma(p) \int_0^\infty \frac{dx x}{(x^2+1)^{p/2}(2\kappa l)^p} \\
&\quad \times \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \\
&= 2(\kappa l)^2 \int_0^\infty dx x \sum_{n=1}^{\infty} n^{-1} e^{-2\kappa l n\sqrt{x^2+1}} \\
&= -2(\kappa l)^2 \int_0^\infty dx x \ln\left(1 - e^{-2\kappa l\sqrt{x^2+1}}\right) \\
&= -2l^2 \int_\kappa^\infty dt t \ln\left(1 - e^{-2lt}\right). \qquad \text{(B.49)}
\end{aligned}$$

The free energy from η_1 is then seen to give a contribution equal to the zero frequency part of the Lifshitz-Casimir energy between ideal metal surfaces with an intervening plasma [30],

$$F_1(l,T) = \frac{kT}{2\pi} \int_\kappa^\infty dt t \ln\left(1 - e^{-2lt}\right). \qquad \text{(B.50)}$$

The remaining η term is

$$\eta_2 \approx 2\pi x^3 \int_1^{x^2} dy y^{-5/2} e^{-\pi\bar{\rho}y} \bar{\omega}(y) \bar{\omega}\left(x^2/y\right). \qquad \text{(B.51)}$$

Now, since

$$\bar{\omega}(y) = \sum_{n=1}^{\infty} e^{-n^2\pi y} = \sum_{n=0}^{\infty} e^{-\pi(n+1)^2 y}, \qquad \text{(B.52)}$$

we have

$$e^{-\pi y} < \bar{\omega}(y) < \frac{e^{-\pi y}}{1 - e^{-2\pi y}}, \qquad \text{(B.53)}$$

and

$$\begin{aligned}
&\int_1^{x^2} dy y^{-5/2} e^{-\pi\bar{\rho}y} e^{-\pi y} e^{-\pi x^2/y} < I_2 \\
&\quad < \int_1^{x^2} dy y^{-5/2} \frac{e^{-\pi\bar{\rho}y} e^{-\pi y} e^{-\pi x^2/y}}{(1 - e^{-2\pi y})(1 - e^{-2\pi x^2/y})} \\
&\quad < \int_1^{x^2} dy y^{-5/2} \frac{e^{-\pi\bar{\rho}y} e^{-\pi y} e^{-\pi x^2/y}}{\left(1 - e^{-2\pi}\right)^2}. \qquad \text{(B.54)}
\end{aligned}$$

Apart from a very small uncertainty $(1-e^{-2\pi})^2$ we have:

$$\eta_2 \approx 2\pi x^3 \int_1^{x^2} dy y^{-5/2} e^{-\pi\bar{\rho}y} e^{-\pi y} e^{-\pi x^2/y}, \qquad \text{(B.55)}$$

and with the substitution $y \to yx$ we have

$$\approx 2\pi x^{3/2} \int_{1/x}^{x} dy y^{-5/2} e^{-\pi\bar{\rho}xy} e^{-\pi(y+1/y)x}, \qquad \text{(B.56)}$$

which for $x \to \infty$ (large separations or high temperatures) produces a simple final expression. To find this we notice that the integral has a steep maximum. Take $f(y) = y + 1/y$, then $f'(y) = 1 - 1/y^2$ is equal to zero at $y_0 = 1$ and $f(y_0) = 2$ and $f''(y_0) = 2$. Thus, we may write

$$\eta_2 \approx 2\pi x^{3/2} \int_{-\infty}^{\infty} dy e^{-\pi\bar{\rho}x} e^{-2\pi x} e^{-\pi x(y-y_0)^2}, \qquad \text{(B.57)}$$

and

$$\eta_2 \approx 4\pi x^{3/2} e^{-\pi\bar{\rho}x} e^{-2\pi x} \int_0^\infty dt e^{-\pi x t^2} = 2\pi x e^{-\pi\bar{\rho}x} e^{-2\pi x}. \qquad \text{(B.58)}$$

The free energy from η_2 gives a contribution at high x (large separations or high temperatures):

$$\begin{aligned}
F_2 &= \frac{-(kT)^2}{l\hbar c} e^{-\bar{\rho}x} e^{-2\pi x} \\
&= \frac{-(kT)^2}{l\hbar c} e^{-2\rho\hbar c \frac{e^2}{mc^2} l} e^{-4\pi kTl/(\hbar c)}. \qquad \text{(B.59)}
\end{aligned}$$

The whole Casimir free energy in the high $x = 2kTl/(\hbar c)$ limit is:

$$\begin{aligned}
F(l,T) &= \frac{kT}{2\pi} \int_\kappa^\infty dt t \ln\left(1 - e^{-2lt}\right) \\
&\quad - \frac{(kT)^2}{l\hbar c} e^{-2\rho\hbar c \frac{e^2}{mc^2} l} e^{-4\pi kTl/(\hbar c)} + O\left(e^{-x^2}\right). \qquad \text{(B.60)}
\end{aligned}$$

This is the correct limit for either high temperature at fixed separation or for large distances at fixed temperature. The given expression can also be valid at small separations or low temperatures. This is a crucial point but one should remember that the derivation of plasma density from the equating of black body radiation to zero-point energy and subsequent use of that density requires "high" temperatures [28]. The situation for two nuclear particles is one with very high effective temperature and separations being "large", at least compared to the screening length of the high density plasma.

Appendix C: ζ functions in physics

We would like to point out that that zeta functions have been applied to many physical problems in the past [38–42]. Elizalde considered for example the sum $S_2(t)$, defined by:

$$S_2(t) = \sum_{n=1}^{\infty} e^{-n^2 t},$$

with t a parameter. This is transformed into the equation

$$S_2(t) = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\pi}{t}} + \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} \zeta(-2k) + \Delta_2(t),$$

where $\Delta_2(t)$ is a remainder. The zeta-function term does not contribute, and the reminder reduces to the sum/integral

$$\Delta_2(t) = 2\sum_{n=1}^{\infty}\int_0^{\infty} dx e^{-x^2 t}\cos(2\pi nx) = \sqrt{\frac{\pi}{t}}\sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2}{t}}.$$

It means that

$$S_2(t) = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\pi}{t}} + \sqrt{\frac{\pi}{t}}\sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2}{t}}.$$

This formula was a key component in our derivations [35].

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